Nearly incompressible fluids: Hydrodynamics and large scale inhomogeneity

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(Received 15 February 2006; published 16 August 2006)

A system of hydrodynamic equations in the presence of large-scale inhomogeneities for a high plasma beta solar wind is derived. The theory is derived under the assumption of low turbulent Mach number and is developed for the flows where the usual incompressible description is not satisfactory and a full compressible treatment is too complex for any analytical studies. When the effects of compressibility are incorporated only weakly, a new description, referred to as "nearly incompressible hydrodynamics," is obtained. The nearly incompressible theory, was originally applied to homogeneous flows. However, large-scale gradients in density, pressure, temperature, etc., are typical in the solar wind and it was unclear how inhomogeneities would affect the usual incompressible and nearly incompressible descriptions. In the homogeneous case, the lowest order expansion of the fully compressible equations leads to the usual incompressible equations, followed at higher orders by the nearly incompressible equations, as introduced by Zank and Matthaeus. With this work we show that the inclusion of large-scale inhomogeneities (in this case time-independent and radially symmetric background solar wind) modifies the leading-order incompressible description of solar wind flow. We find, for example, that the divergence of velocity fluctuations is nonsolenoidal and that density fluctuations can be described to leading order as a passive scalar. Locally (for small lengthscales), this system of equations converges to the usual incompressible equations and we therefore use the term "locally incompressible" to describe the equations. This term should be distinguished from the term "nearly incompressible," which is reserved for higher-order corrections. Furthermore, we find that density fluctuations scale with Mach number linearly, in contrast to the original homogeneous nearly incompressible theory, in which density fluctuations scale with the square of Mach number. Inhomogeneous nearly incompressible equations for higher order fluctuation components are derived and it is shown that they converge to the usual homogeneous nearly incompressible equations in the limit of no large-scale background. We use a time and length scale separation procedure to obtain wave equations for the acoustic pressure and velocity perturbations propagating on fast-time–short-wavelength scales. On these scales, the pseudosound relation, used to relate density and pressure fluctuations, is also obtained. In both cases, the speed of propagation (sound speed) depends on background variables and therefore varies spatially. For slow-time scales, a simple pseudosound relation cannot be obtained and density and pressure fluctuations are implicitly related through a relation which can be solved only numerically. Subject to some simplifications, a generalized inhomogeneous pseudosound relation is derived. With this paper, we extend the theory of nearly incompressible hydrodynamics to flows, including the solar wind, which include large-scale inhomogeneities (in this case radially symmetric and in equilibrium).

DOI: 10.1103/PhysRevE.74.026302 PACS number(s): 47.40.Dc, 96.50.Ci, 52.35.Ra, 47.27.−i

I. INTRODUCTION

Hydrodynamic and magnetohydrodynamic models are widely used and applied in many areas of physics. Depending on the physical properties of the system, one often has to decide on either an incompressible or a fully compressible description. The incompressible description has proved useful for those problems where the effects of low-frequency turbulence are particularly important, and for which it is sufficient to obtain the description of the system at the lowest order or when computer resources are inadequate (e.g., high Reynolds number simulations). The fully compressible description is an exact description of the fluid, but it is often too analytically and computationally arduous and an incompressible description is chosen instead just for its mathematical tractability. This, however, sometimes oversimplifies the problem and relationship to compressibility is lost. In many cases (especially when the density fluctuations are small compared to their mean density value) we would like not to exclude the effects of compressibility completely, but admit them only weakly. A theory where compressible effects are incorporated only weakly was developed by Klainerman and Majda [1,2], Zank, Matthaeus, and Brown [3–6] and is usually referred as a "nearly incompressible theory." The nearly incompressible (NI) theory introduces a third possible description in addition to usual incompressible/compressible choices and under some assumptions represents a bridge between these two formalisms, namely in the limit of low Mach number, where the solutions of compressible and incompressible regimes converge.

An excellent example of a fully developed turbulent magnetohydrodynamical (MHD) fluid is the solar wind flow, and models based on an incompressible MHD description have traditionally been used for over 40 years. These models have yielded considerable success in modeling and explaining much observational data, such as, for instance, the Kolmogorov-like power spectrum observed in magnetic field fluctuations. Solar wind observations exhibit a Kolmogorov-like spectrum for low-frequency velocity, and magnetic energy fluctuations, and the simplest explanation of the observations is based on incompressible MHD fluid theories which assume isotropy and homogeneity of the flow (Kol-

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mogorov [7], Kraichnan [8], Lesieur [9], Biskamp [10], and Oughton [11]). Rather remarkably, density fluctuations, both observed in situ in the solar wind and by radiowave scintillation measurements in the interplanetary medium (IPM) and interstellar medium (ISM) also exhibit a Kolmogorov-like spectrum (Armstrong *et al.* [12], Matthaeus *et al.* [13]). An explanation for the density fluctuation spectrum observed in the ISM was based on an incompressible MHD fluid description (Montgomery *et al.* [14]), who suggested that density fluctuations are proportional to pressure fluctuations ($\delta p = c_s^2 \delta\rho$), where the constant of proportionality is the square of the sound speed. This is usually referred as a *pseudosound* approximation. The relation was based on the assumption of very large sound speeds (i.e., the incompressible regime) and using an adiabatic equation of state (although the pressure was determined by solving the incompressible MHD equations), and a $k^{-5/3}$ spectral law, resulted as a direct consequence of assuming a $k^{-5/3}$ magnetic energy spectrum. This model can therefor explain the density fluctuations only in the presence of magnetic fields and not in pure hydrodynamic flows. According to measurements, the density fluctuations deviate from its mean value by about 10% (Matthaeus *et al.* [13], Spangler [15]). This suggests that density fluctuations in the interstellar medium and solar wind (SW) are only weakly compressible, but sufficiently large enough, that an incompressible description is inappropriate.

Isotropic homogeneous incompressible and compressible turbulence models for the solar wind can be used as long as there are no strong inhomogeneities present in the background flow. However, there are many cases, where large-scale gradients in physical variables such as density, pressure, velocity, and magnetic fields are present in the background flow. These gradients can act as additional sources of fluctuations and change the behavior of the system dramatically. This can happen, for example, when characteristic turbulent fluctuation lengthscales are comparable with the gradients in the background flow (e.g., Zank [16]). Our primary goal in this work is therefore to include large-scale inhomogeneities to the theory of the turbulence of the solar wind.

Bhattacharjee, Ng, and Spangler [17] developed a model for weakly compressible MHD turbulence in the solar wind in which they included spatial inhomogeneity in the background magnetic field, obtaining a reduced MHD model involving eight scalar variables, described as a four-field model. The basic variables were magnetic flux, vorticity, pressure and parallel flow in the leading order weakly compressible regime. This model is not as simple as reduced MHD (developed by Rosenbluth *et al.* [18], Strauss [19], and Zank and Matthaeus [6]), which is a two-field model. However, because of the greater generality associated with more variables and mainly because of the inclusion of large scale inhomogeneities, it seems to describe effects of higher order and complexity in MHD turbulence of solar wind, which were not possible to show by the simpler two-field reduced MHD model. Inclusion of large-scale inhomogeneities therefore looks very promising and is a necessary step for the correct description of the solar wind turbulence.

Nearly incompressible fluid theory, as developed by Zank, Matthaeus, and Brown, is an appropriate model to describe weakly compressible solar wind fluctuations. The theory was developed primarily for spatial, homogeneous fluctuations in the solar wind. However, as mentioned above, the solar wind possesses large-scale background gradients and in order to describe these fluctuations in a self consistent manner, it is necessary to include large-scale background inhomogeneities in the NI model. This modifies the nearly incompressible description, introducing subtle nonlinear effects.

The nearly incompressible theory describes the expansion of the fully compressible fluid equations (including MHD) in terms of weaker fluctuations compared to the incompressible background. The expansion parameter is essentially the turbulent Mach number, and therefore NI theory primarily describes flows for which fluctuations are in low Mach number regimes. In the expansion, the leading order background terms satisfy the incompressible equations, while the higher order fluctuations yield a weakly compressible set of equations. Zank and Matthaeus [4–6] derived an extensive theory for nearly incompressible fluid dynamics and magnetohydrodynamics, including thermal conduction and a nonadiabatic equation of state. The predictions of NI MHD were found to hold for a rich variety of solar wind observations (Zank *et al.* [20], Matthaeus *et al.* [13], and Klein *et al.* [21]). However, NI theory has so far neglected the inclusion of large-scale background inhomogeneities. Homogeneity in the NI theory lead to the original Zank and Matthaeus prediction that pressure and density fluctuations must be scale with the square of the Mach number. This was a source of criticism for the NI theory (Tu and Marsch [22] and Bavassano, Bruno, and Klein [23]), since SW observations frequently suggested a linear scaling with turbulent Mach number. As we show in this paper, the inclusion of large-scale inhomogeneities addresses this criticism in the NI theory, and at the leading order linear Mach number density fluctuations are present.

The hydrodynamical NI equations, without either large scale inhomogeneities or viscous terms, were first given by Klainerman and Majda [1,2], who rigorously proved, using functional analytic techniques and appropriate assumptions, that the solutions of the NI equations converge to the solutions of the standard incompressible equations as the sound Mach number becomes small. Klainerman and Majda (hereafter referred to as KM) in these two papers proved several theorems which are of great importance for hydrodynamics and especially for nearly incompressible theory. We do not reproduce their results here, but because these theorems represents the rigorous justification for nearly incompressible theory we briefly summarize them. The theory was developed in the general framework of a quasilinear hyperbolic system of partial differential equations, defined for the usual $x \in R^N$ or in the periodic case for simplicity on an N-dimensional torus. Distances between functions are defined by appropriate maximum norms on the square integrable (L^2) Sobolev spaces of order s, denoted by H^s (because for an appropriately defined norm, this space is a Hilbert space).

The first theorem addresses the uniform stability of compressible solutions in the incompressible (low Mach number M) limit and shows that with prescribed incompressible initial data, there exists a finite time interval T, independent of M, such that a unique classical C^1 solution (space of continuous functions with continuous first derivatives) of incompressible fluid equations exists for all $M \to 0$ on the interval $[0,T]$, and additional inequalities for the "closeness" of solutions are also satisfied. The same results hold when small fluctuations to the incompressible initial data are added. After introducing the additional condition that the initial fluctuations are of the order of Mach number $O(M)$ for velocity fluctuations and order Mach number squared $O(M^2)$ for pressure fluctuations and, of course, assuming that both sets of fluctuations are bounded, the theorem states that, on the same time interval $[0,T]$, the velocity solution of the compressible equations converges to the velocity solution of the incompressible equations as $M \to 0$. This convergence is weak in L^∞ space (essentially bounded functions) and uniform in $C^1_{\rm loc}$ space (locally C^1). KM also show that, when the initial data for velocity and pressure fluctuations is more general, then as $M \to 0$, the velocity solution of the compressible equations still converges weakly in L^∞ to some velocity $\tilde{v}$, but, because there is insufficient compactness in time in the limiting process $M \to 0$, we cannot conclude that this $\tilde{v}$ is the solution of the incompressible equations. The compactness in time is recovered only when initial data for fluctuations scale with Mach number as stated above. The first theorem was also generalized to the Navier-Stokes equations where viscosity is present.

The second theorem consists of three parts, and each part assumes boundedness of the incompressible pressure together with its first derivative. In the first part, KM show the long time existence for the first theorem, namely, that for arbitrary long T_0, there exists a sufficiently small Mach number M, that the first theorem is valid over the time interval $[0,T_0]$. In the second part, KM provide a justification for linearized acoustics based on the nearly incompressible equations, which contain first order corrections in pressure and velocity. Finally in the third part, KM prove that for the compressible equations there exists a complete convergent expansion series in powers of small Mach number. This, together with other results of KM [24], provides an important proof that the ideas of nearly incompressible theory are mathematically well defined and correct. A few related theorems were also obtained for MHD, but we do not discuss them here. Even though we do not develop our work in the rigorous tradition of Klainerman and Majda, their results provide a strong base for us to further generalize the nearly incompressible theory by including other effects such as viscosity, heat conduction, or magnetic fields, as was done by Zank and Matthaeus [5,6].

In this paper, we include a large-scale inhomogeneous background to the nearly incompressible theory. The primary application is assumed to be the solar wind flow, where the background is considered to be radially symmetric and in equilibrium (time independent). We restrict our attention to pure hydrodynamics, but the results are also valid without any loss of generality to the high beta MHD regime. The generalization to MHD, together with numerical simulations, will be the subject of a subsequent paper. The general formalism and notation was developed by Zank and Matthaeus [5], and the present analysis is a direct extension of that article.

To avoid later confusion, we should clearly emphasize the distinction between the terms "locally incompressible" (LI) and "nearly incompressible" (NI), which are used throughout the paper. Of course, for both the homogeneous and inhomogeneous cases, we assume that the density, pressure and velocity variations are only small and that the expansion with respect to the Mach number is mathematically justified. In the homogeneous case, when the normalized compressible equations are expanded, we obtain at the lowest-order the usual incompressible equations. The set of equations obtained at the lowest-order from the fully compressible equations is sometimes referred as the leading-order incompressible description. Continuing to higher orders, we obtain the "nearly incompressible" system of equations, as shown by Zank and Matthaeus [5] and Klainerman and Majda [24]. However, when large-scale inhomogeneities are correctly included in the compressible equations and the appropriate expansion procedure is followed, we obtain at the lowest order a system of equations that is not incompressible, but includes small compressible effects (e.g., the divergence of the velocity is small, but nonzero; density variations are present). Because the term "nearly incompressible" has been used for many years and is reserved in the literature to describe higher-order corrections to the core incompressible solutions, we have chosen to describe the derived lowest-order inhomogeneous equations by the term "locally incompressible." This is motivated by the fact that, when a lengthscale separation is introduced, we find that on small lengthscales (locally), the system of locally incompressible equations converges to the usual incompressible equations. In summary, with the inclusion of large-scale inhomogeneities, the leading order incompressible description is called locally incompressible. For the higher-order inhomogeneous equations we naturally use the term "inhomogeneous nearly incompressible."

The paper is organized as follows. The basic hydrodynamical equations with the inclusion of large-scale inhomogeneities are presented in Sec. II, followed by their normalization in Sec. III. Sec. IV introduces the locally incompressible expansions together with time and wavelength scale separation. In Sec. V, the higher-order inhomogeneous nearly incompressible equations are derived and their physical meaning is discussed. Section VI derives a generalized pseudosound relation. Finally, in Sec. VII, we summarize our main results and in the Appendix, for convenience, we present the non-normalized, incompressible, and nearly incompressible inhomogeneous equations.

II. BASIC EQUATIONS

The hydrodynamic equations in the absence of viscosity, heat conduction, sources/sinks and forces are as usual

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0, \tag{1}$$

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p, \tag{2}$$

$$\frac{\partial p}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} p + \gamma p \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0. \tag{3}$$

The internal energy equation for an adiabatic (ideal fluid) is of course equivalent to

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}. \tag{4}$$

In applying Eqs. (1)–(4) to a large scale inhomogeneous flow such as the solar wind, we assume that the background flow is both spherically symmetric and time independent, i.e., that ρ_{SW}, $\boldsymbol{u}_{\mathrm{SW}}$, and p_{SW} depend only on the spatial spherical coordinate r, with the usual definition of the spherical coordinates $(\hat{r}, \hat{\theta}, \hat{\phi})$. So for example the velocity of solar wind is $\boldsymbol{u}_{\mathrm{SW}} = (u_{\mathrm{SW}}, 0, 0)$. Then, the steady-state background flow satisfies

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho_{\mathrm{SW}} u_{\mathrm{SW}}) = 0, \tag{5}$$

$$\rho_{\mathrm{SW}} u_{\mathrm{SW}} \frac{\partial u_{\mathrm{SW}}}{\partial r} = -\frac{\partial p_{\mathrm{SW}}}{\partial r}. \tag{6}$$

Consider now fluctuations in the background spherically symmetric solar wind, and regard the inhomogeneous background as the mean field. Since we made no assumptions about the symmetry of the fluctuations, we express the fluctuations in Cartesian coordinates. We use a mixed coordinate system, where large scale solar wind variations depend only on r and the small scale fluctuations on Cartesian coordinates. The mean field expansion of the flow variables in the presence of small scale SW fluctuations can be described as follows:

$$\boldsymbol{u} = \boldsymbol{u}_{\mathrm{SW}}(r) + \boldsymbol{u}'(x,y,z,t),$$

$$\rho = \rho_{\mathrm{SW}}(r) + \rho'(x,y,z,t),$$

$$p = p_{\mathrm{SW}}(r) + p'(x,y,z,t).$$

Equations (1)–(3) can then be expanded as

$$\frac{\partial \rho'}{\partial t} + (\boldsymbol{u}_{\mathrm{SW}} + \boldsymbol{u}') \cdot \boldsymbol{\nabla}\rho' + (\rho_{\mathrm{SW}} + \rho')\boldsymbol{\nabla} \cdot \boldsymbol{u}' = -\boldsymbol{u}' \cdot \boldsymbol{\nabla}\rho_{\mathrm{SW}} - \rho' \boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathrm{SW}}, \tag{7}$$

$$(\rho_{\mathrm{SW}} + \rho')\frac{\partial \boldsymbol{u}'}{\partial t} + (\rho_{\mathrm{SW}} + \rho')\boldsymbol{u}_{\mathrm{SW}} \cdot \boldsymbol{\nabla}\boldsymbol{u}' + (\rho_{\mathrm{SW}} + \rho')\boldsymbol{u}' \cdot \boldsymbol{\nabla}\boldsymbol{u}' = -\boldsymbol{\nabla}p' - \rho' \boldsymbol{u}_{\mathrm{SW}} \cdot \boldsymbol{\nabla}\boldsymbol{u}_{\mathrm{SW}} - (\rho_{\mathrm{SW}} + \rho')\boldsymbol{u}' \cdot \boldsymbol{\nabla}\boldsymbol{u}_{\mathrm{SW}}, \tag{8}$$

$$\frac{\partial p'}{\partial t} + \boldsymbol{u}_{\mathrm{SW}} \cdot \boldsymbol{\nabla}p' + \boldsymbol{u}' \cdot \boldsymbol{\nabla}p' + \gamma(p_{\mathrm{SW}} + p')\boldsymbol{\nabla} \cdot \boldsymbol{u}' = -\boldsymbol{u}' \cdot \boldsymbol{\nabla}p_{\mathrm{SW}} - \gamma p' \boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathrm{SW}}. \tag{9}$$

Equations (7)–(9) describe the evolution of the small scale fluctuating density, velocity, and pressure, respectively, in the presence of a large scale static equilibrium background. The background equilibrium flow is treated as nonevolutionary in our treatment, and therefore does not self-consistently feed back on the fluctuations. An active feedback of the large-scale equilibrium on the small scale turbulence fluctuations requires a completely self-consistent treatment, which is beyond the scope of our current analysis. Equations (7)–(9) behave linearly when the background is much stronger than the small-scale turbulence fluctuations. However, in the regime where these fluctuations are comparable with the background flow amplitude, nonlinear interactions become important. The nonlinear solution of such a problem would then be of importance to understand how turbulence is mediated by large scale flows.

III. NORMALIZATIONS

Scintillation measurements of radio waves in ISM and direct measurements in the SW show that typical density fluctuations are of the order of 10% of the mean density field (for example Matthaeus *et al.* [13] and Spangler [15]). To develop the reduction to a leading-order incompressible description from Eqs. (7)–(9), it is necessary to introduce appropriate normalizations. Let ρ_0, u_0, p_0 be characteristic fixed parameters for solar wind fluctuations. The normalized quantities are written as

$$\tilde{\rho}_{\mathrm{SW}} = \frac{\rho_{\mathrm{SW}}}{\rho_0}, \quad \tilde{\boldsymbol{u}}_{\mathrm{SW}} = \frac{\boldsymbol{u}_{\mathrm{SW}}}{u_0}, \quad \tilde{p}_{\mathrm{SW}} = \frac{p_{\mathrm{SW}}}{p_0},$$

$$\tilde{\rho}' = \frac{\rho'}{\rho_0}, \quad \tilde{u}' = \frac{\boldsymbol{u}'}{u_0}, \quad \tilde{p}' = \frac{p'}{p_0}.$$

Furthermore, let R and L be typical lengthscales for the large scale solar wind and the fluctuations, respectively, and introduce the following normalizations:

$$\boldsymbol{x}^* = \frac{\boldsymbol{x}}{L}, \quad t^* = \frac{u_0}{L}t, \quad r^* = \frac{r}{R},$$

$$\frac{\partial}{\partial x} = \frac{1}{L}\frac{\partial}{\partial x^*} \to \boldsymbol{\nabla}_x = \frac{1}{L}\boldsymbol{\nabla}^*_x,$$

$$\frac{\partial}{\partial r} = \frac{1}{R}\frac{\partial}{\partial r^*} \to \boldsymbol{\nabla}_r = \frac{1}{R}\boldsymbol{\nabla}^*_r,$$

$$\frac{\partial}{\partial t} = \frac{u_0}{L}\frac{\partial}{\partial t^*}.$$

We introduce the dimensionless parameter $\chi = L/R$, which is the ratio of characteristic small to large scale lengths. Typically, we can take for large-scale solar wind lengthscales $R = 0.1-1$ AU and for characteristic small-scale fluctuations we can use the correlation length, which is typically $L = 0.01$ AU, giving us at most $\chi = 0.1 \ll 1$, and we will use this later. We can rewrite the continuity Eq. (7) in the normalized form

$$\frac{\partial \tilde{\rho}'}{\partial t^*} + (\tilde{\boldsymbol{u}}_{\mathrm{SW}} + \tilde{\boldsymbol{u}}') \cdot \boldsymbol{\nabla}^*_x\tilde{\rho}' + (\tilde{\rho}_{\mathrm{SW}} + \tilde{\rho}')\boldsymbol{\nabla}^*_x \cdot \tilde{\boldsymbol{u}}' = -\chi\tilde{\boldsymbol{u}}' \cdot \boldsymbol{\nabla}^*_r\tilde{\rho}_{\mathrm{SW}} - \chi\tilde{\rho}'\boldsymbol{\nabla}^*_r \cdot \tilde{\boldsymbol{u}}_{\mathrm{SW}}. \tag{10}$$

The parameter χ is of great importance, because it strongly couples the background equilibrium to the fluctuating components and leads to strong nonlinear interactions between the disparate lengthscales. On introducing the usual gas Mach number M_{s0} and gas sound speed c_{s0}

$$M_{s0} = \frac{u_0}{c_{s0}}, \quad c_{s0}^2 = \frac{\gamma p_0}{\rho_0}, \tag{11}$$

we introduce a new parameter $\epsilon^2 = \frac{\rho_0 u_0^2}{p_0} \equiv \gamma M_{s0}^2$. Since the solar wind is predominantly incompressible, the sound speed is typically very large compared to the small-scale characteristic velocity fluctuations. Thus, for typical SW parameters $u_0 \ll c_{s0}$. This consequently leads to a smaller turbulent Mach number and hence ϵ, the smallness parameter, is much smaller than the unity. The value of the ambient Mach number can be estimated from typical SW parameters. For instance, the solar wind background flow, at nearly 1 AU, is typically $u_{\mathrm{SW}}=468$ km/s, the sound speed defined as $c_s^2 = \frac{\gamma p_{\mathrm{SW}}}{\rho_{\mathrm{SW}}}$ is typically $c_s=63$ km/s, and so the Mach number of the SW flow $M_s = u_{\mathrm{SW}}/c_s = 468/63 = 7.4$. By contrast, velocity fluctuations have a mean value $u_0=10$ km/s. Thus for $c_{s0} \approx c_s$, the turbulent Mach number $M_{s0} \approx 10/63 = 0.16 \ll 1$. This justifies our assumption of low Mach number. The normalized momentum equation can then be expressed as

$$(\tilde{\rho}_{\mathrm{SW}} + \tilde{\rho}')\frac{\partial \tilde{\boldsymbol{u}}'}{\partial t^*} + (\tilde{\rho}_{\mathrm{SW}} + \tilde{\rho}')\tilde{\boldsymbol{u}}_{\mathrm{SW}} \cdot \boldsymbol{\nabla}_x^* \tilde{\boldsymbol{u}}' + (\tilde{\rho}_{\mathrm{SW}} + \tilde{\rho}')\tilde{\boldsymbol{u}}' \cdot \boldsymbol{\nabla}_x^* \tilde{\boldsymbol{u}}'$$
$$= -\frac{1}{\epsilon^2}\boldsymbol{\nabla}_x^* \tilde{p}' - \chi \tilde{\rho}' \tilde{\boldsymbol{u}}_{\mathrm{SW}} \cdot \boldsymbol{\nabla}_r^* \tilde{\boldsymbol{u}}_{\mathrm{SW}} - (\tilde{\rho}_{\mathrm{SW}} + \tilde{\rho}')\chi \tilde{\boldsymbol{u}}' \cdot \boldsymbol{\nabla}_r^* \tilde{\boldsymbol{u}}_{\mathrm{SW}}, \tag{12}$$

and the normalized energy equation as

$$\frac{\partial \tilde{p}'}{\partial t^*} + (\tilde{\boldsymbol{u}}_{\mathrm{SW}} + \tilde{\boldsymbol{u}}') \cdot \boldsymbol{\nabla}_x^* \tilde{p}' + \gamma(\tilde{p}_{\mathrm{SW}} + \tilde{p}')\boldsymbol{\nabla}_x^* \cdot \tilde{\boldsymbol{u}}'$$
$$= -\chi \tilde{\boldsymbol{u}}' \cdot \boldsymbol{\nabla}_r^* \tilde{p}_{\mathrm{SW}} - \gamma \chi \tilde{p}' \boldsymbol{\nabla}_r^* \cdot \tilde{\boldsymbol{u}}_{\mathrm{SW}}. \tag{13}$$

The adiabatic expression for the large scale solar wind $\tilde{p}_{\mathrm{SW}} = \tilde{\rho}_{\mathrm{SW}}^{\gamma}$ gives us the useful relation

$$\frac{\boldsymbol{\nabla}\tilde{\rho}_{\mathrm{SW}}}{\tilde{\rho}_{\mathrm{SW}}} = \frac{1}{\gamma}\frac{\boldsymbol{\nabla}\tilde{p}_{\mathrm{SW}}}{\tilde{p}_{\mathrm{SW}}}, \tag{14}$$

which will be used below. Equations (10), (12), and (13) describe respectively the normalized fluctuating density, velocity, and the pressure in the solar wind plasma. It is noteworthy here that the right-hand sides of these equations, due to the large-scale gradients of the stationary equilibrium solar flows, behave as sources. These sources therefore disappear in the absence of flow gradients. Similarly, the dimensionless parameter χ is an important quantity that describes the coupling between the large-scale equilibrium flow and small-scale turbulence. This parameter therefore emerges naturally from the terms that contain SW equilibrium gradients in Eqs. (10)–(13). The coupling parameter χ introduces instabilities due to free energy associated with various gradients. Typically, $\chi \ll 1$ because the equilibrium flow varies much more slowly than the fluctuations in the solar wind plasma. In the limit of negligibly small χ, the solar wind plasma is predominantly composed of a small-scale high-frequency component and contains no inhomogeneous flows. Such a trivial limit recovers the basic homogeneous hydrodynamic equations.

A leading-order incompressible description of the solar wind plasma can also be deduced from Eqs. (10)–(14), using the perturbative expansion method developed by Zank and Matthaeus [4,5]. This method uses a multiple time expansion technique that distinctively separates fast and slow scales and is described in detail in the subsequent section.

IV. LOCALLY INCOMPRESSIBLE EQUATIONS

To derive a locally incompressible system of equations from the normalized, fully compressible system (7)–(9), we follow the constructive approach developed by Zank and Matthaeus (ZM) who applied a procedure developed by Kreiss. Kreiss [25] showed that high frequency fluctuations in a hyperbolic system of equations could be eliminated by assuming boundedness of several orders of time derivatives. Such a procedure has the effect of implying further constraints on the system of equations. ZM showed explicitly that bounding the time derivatives of the compressible hydrodynamic equations, according to Kreiss' [25] principle, yielded the familiar equations of the incompressible fluid mechanics. As ZM observed, the physical argument advanced typically to justify the use of the incompressible fluid description (e.g., Landau and Lifshitz [26]) is completely consistent with the mathematical justification, but the latter provides a framework for deriving leading-order incompressible model systems for more complicated problems of homogeneous gas flow (e.g., Zank [27], Florinski *et al.* [28]). Because we allow variation of the density at the lowest order, to prevent confusion with the term "incompressible," which usually means that no density change is allowed, we use the term "locally incompressible" (LI) instead. The choice of this term will become more apparent when we introduce a separation of time and length scales at the end of this section.

Following Zank and Matthaeus, we use Kreiss' [25] approach to derive a LI reduction of the compressible hydrodynamic equations in the presence of a large-scale background flow field. In particular, we derive a system of equations that describe the evolution and coupling of small-scale (relative to the background inhomogeneity) LI fluctuations to the expanding solar wind (or stellar wind, in general). To derive LI equations, we introduce an ordering expansion for the fluctuating quantities according to

$$\tilde{\rho}' = \epsilon \rho^{\infty}, \quad \tilde{p}' = \epsilon p_1 + \epsilon^2 p^{\infty}, \quad \tilde{\boldsymbol{u}}' = \boldsymbol{u}^{\infty}. \tag{15}$$

The motivation for choosing this expansion is to separate different magnitudes in solar wind and fluctuation quantities. This is done using the small parameter ϵ. In the leading order, the total density is composed of a background solar wind density ρ_{SW} plus a correction for the small density fluctuation $\epsilon\rho^{\infty}$. The magnitude of the fluctuating velocity $\boldsymbol{u}^{\infty}$ is assumed to be of similar order as the background solar wind speed $\boldsymbol{u}_{\mathrm{SW}}$. The expansion in $\tilde{p}$ is very critical and must be dealt with carefully as it leads to high frequency oscillations in the momentum equation. Note that p_1 appears at an order ϵ, whereas p^{∞} because of Kreiss' principle, at an order ϵ^2.

The expansion is chosen to be consistent with the earlier ZM analysis without the solar wind background [4–6]. For now, we assume that the parameter χ is of arbitrary order.

We use Eq. (15) in the normalized equations (10), (12), and (13), and collect terms of the same order in the small parameter ϵ. For simplicity we drop the index x in the nabla operator and the symbols $*,\sim$. Because $\epsilon=\gamma^{1/2}M_{s0}$, the expansion is only for low turbulent Mach numbers to ensure that the expansion remains valid and does not lead to nonphysically growing solutions.

Using Kreiss' principle in the momentum equation (12) implies that to prevent the acceleration $\frac{\partial \mathbf{u}^\infty}{\partial t}$ being unbounded, the term at the order $\frac{1}{\epsilon}$ must be equal to zero. This yields a constraint

$$O\left(\frac{1}{\epsilon}\right):\quad \nabla p_1 = 0, \tag{16}$$

meaning that p_1 is constant throughout space and that first order pressure fluctuations are not present. We can therefor exclude first order pressure p_1 from all expansions and instead of $\tilde{p}=p_{\mathrm{SW}}+\epsilon p_1+\epsilon^2 p^\infty$ we will use $\tilde{p}=p_{\mathrm{SW}}+\epsilon^2 p^\infty$. The same conclusion to exclude ϵ-order pressure fluctuations was obtained without a large-scale background by Zank and Matthaeus [5] and by Matthaeus and Brown [3]. Matthaeus and Brown used a different procedure, where time-scale separation was used (without length-scale separation) directly in the compressible equations (10)–(13) and a sourceless wave equation for p_1 propagating on fast time scales was obtained. They argue that if such waves are present and viscous dissipation is reinstated, then they decay through viscous damping and if the waves are absent in the initial data, they cannot appear since they have no sources. However, they conclude that the main reason for the exclusion of p_1 pressure variations is that they would produce unbounded velocity perturbations at vanishing Mach number. To better understand the nearly incompressible model, we performed a thorough investigation of what the consequences of a nonvanishing first order pressure p_1 would be. We concluded, that at least in the hydrodynamics case, the inclusion of p_1 creates subsequent inconsistencies in the nearly incompressible description. For example, it leads to a problem in defining the leading order incompressible equations (e.g., in energy equation for p_1), and also induces an awkward nonphysical singularity in the generalized pseudosound relation. The procedure used by Matthaeus and Brown was also checked and a similar sourceless wave equation was obtained. Without presenting all the details, we found from a variety of approaches that to obtain a self-consistent nearly incompressible description, the exclusion of a leading order pressure p_1 is necessary and it might be considered as a general feature of nearly incompressible hydrodynamics. However, one has to be careful, because the situation can be much different in the presence of magnetic fields (Bhattacharjee *et al.* [17]), and we will explore this in a subsequent paper, which will be devoted to magnetohydrodynamics. We summarize the expansion that we use henceforth,

$$\tilde{\rho}' = \epsilon\rho^\infty, \quad \tilde{p}' = \epsilon^2 p^\infty, \quad \tilde{\boldsymbol{u}}' = \boldsymbol{u}^\infty. \tag{17}$$

On continuing with the expansion of the momentum equation and collecting terms at the next order, we obtain

$$O(\epsilon^0):\quad \rho_{\mathrm{SW}}\frac{\partial \boldsymbol{u}^\infty}{\partial t} + \rho_{\mathrm{SW}}(\boldsymbol{u}_{\mathrm{SW}}+\boldsymbol{u}^\infty)\cdot\nabla\boldsymbol{u}^\infty = -\nabla p^\infty - \chi\rho_{\mathrm{SW}}\boldsymbol{u}^\infty\cdot\nabla_r\boldsymbol{u}_{\mathrm{SW}}. \tag{18}$$

The form of Eq. (18) resembles the compressible momentum equation except that ρ_{SW} is in fact prescribed by the background flow, and so Eq. (18) is really analogous to the incompressible momentum equation instead. From the continuity equation (10), we have

$$O(\epsilon^0):\ \nabla\cdot\boldsymbol{u}^\infty = -\frac{\chi}{\rho_{\mathrm{SW}}}\boldsymbol{u}^\infty\cdot\nabla_r\rho_{\mathrm{SW}}, \tag{19}$$

which, after using Eq. (14), can be rewritten as

$$O(\epsilon^0):\ \nabla\cdot\boldsymbol{u}^\infty = -\frac{\chi}{\gamma p_{\mathrm{SW}}}\boldsymbol{u}^\infty\cdot\nabla_r p_{\mathrm{SW}}. \tag{20}$$

On collecting terms of higher order yields a passive scalar equation with source terms for the $O(M_{s0})$ density fluctuations

$$O(\epsilon^1):\ \frac{\partial\rho^\infty}{\partial t} + (\boldsymbol{u}_{\mathrm{SW}}+\boldsymbol{u}^\infty)\cdot\nabla\rho^\infty + \rho^\infty\nabla\cdot\boldsymbol{u}^\infty = -\chi\rho^\infty\nabla_r\cdot\boldsymbol{u}_{\mathrm{SW}}, \tag{21}$$

which corresponds to

$$O(\epsilon^1):\ \frac{\partial\rho^\infty}{\partial t} + (\boldsymbol{u}_{\mathrm{SW}}+\boldsymbol{u}^\infty)\cdot\nabla\rho^\infty = \chi\rho^\infty\left(\frac{1}{\rho_{\mathrm{SW}}}\boldsymbol{u}^\infty\cdot\nabla_r\rho_{\mathrm{SW}} - \nabla_r\cdot\boldsymbol{u}_{\mathrm{SW}}\right). \tag{22}$$

Several interesting points are suggested by Eqs. (19) and (22). Unlike the regular incompressible hydrodynamic equations, the velocity fluctuations are nonsolenoidal and a source term induced by the background inhomogeneity is present. The model of Bhattacharjee *et al.* [17], assuming a background large-scale inhomogeneity, does not have the nonsolenoidal condition (20). The nonsolenoidal condition, as we derive below, also introduces an additional source term into the equation for vorticity $\omega=\nabla\times\boldsymbol{u}^\infty$ that arises when eliminating the pressure contribution for the LI momentum equation. Unlike regular incompressible hydrodynamics, the LI density fluctuations are not constant but instead are of $O(M_{s0})$ and respond to the LI flow field $\boldsymbol{u}^\infty$ as a passive scalar while being generated/driven by the low frequency coupling of LI fluctuations $(\boldsymbol{u}^\infty,\rho^\infty)$ to the gradient of the large-scale flow field. The existence of $O(M_{s0})$ density fluctuations is consistent with Bhattacharjee *et al.*, who similarly found that LI fluctuations, coupling to the large-scale inhomogeneous magnetic field, generate $O(M_{s0})$ density fluctuations. As they emphasized, the prediction of $O(M_{s0})$ density scalings is a distinguishing feature of the inhomogeneous model, compared to the homogeneous NI hydrodynamic model which predicts only $O(M_{s0}^2)$ scaling of density fluctuations. Observations of density fluctuations in the SW (Tu and Marsch [22], Bavassano and Bruno [29], Bavassano *et al.* [23]) do not appear to show convincingly that they scale as $O(M_{s0}^2)$, but appear to suggest that there may be a mixture of $O(M_{s0})$ fluctuations as well. However, deriving the Mach number scalings for density fluctuations from SW observations is not unambiguous since the formal expansion is an asymptotic series.

Expanding the energy equation (13), we get at the lowest order

$$O(\epsilon^0):\ \nabla \cdot \boldsymbol{u}^\infty = -\frac{\chi}{\gamma p_{\rm SW}} \boldsymbol{u}^\infty \cdot \nabla_r p_{\rm SW},$$

which was already obtained from the continuity equation (20). This serves as a check on the consistency of our expansion procedure. At the order $O(\epsilon^1)$ we do not have any contributions and because we consider this as the last order at the leading order incompressible description, Eqs. (18), (19), and (22) describe the underlying locally incompressible equations, which, for convenience, we summarize as

$$O(\epsilon^0):\ \nabla \cdot \boldsymbol{u}^\infty = -\frac{\chi}{\rho_{\rm SW}} \boldsymbol{u}^\infty \cdot \nabla_r \rho_{\rm SW}; \tag{23}$$

$$O(\epsilon^0):\ \rho_{\rm SW}\frac{\partial \boldsymbol{u}^\infty}{\partial t} + \rho_{\rm SW}(\boldsymbol{u}_{\rm SW}+\boldsymbol{u}^\infty)\cdot\nabla\boldsymbol{u}^\infty = -\nabla p^\infty - \chi\rho_{\rm SW}\boldsymbol{u}^\infty\cdot\nabla_r\boldsymbol{u}_{\rm SW}; \tag{24}$$

$$O(\epsilon^1):\ \frac{\partial \rho^\infty}{\partial t} + (\boldsymbol{u}_{\rm SW}+\boldsymbol{u}^\infty)\cdot\nabla\rho^\infty = -\chi\rho^\infty\nabla\cdot\boldsymbol{u}_{\rm SW} + \chi\frac{\rho^\infty}{\rho_{\rm SW}}\boldsymbol{u}^\infty\cdot\nabla_r\rho_{\rm SW}. \tag{25}$$

The solar wind variables $\boldsymbol{u}_{\rm SW}, p_{\rm SW}, \rho_{\rm SW}$ are regarded as known, so Eq. (23)–(25) describe the motion of the fluctuating variables $\boldsymbol{u}^\infty, \rho^\infty, p^\infty$ and therefore we have a system of five equations in five unknowns. To model system (23)–(25) numerically, one has to begin with the subsystem (23) and (24) which is a system of four equations in four variables $\boldsymbol{u}^\infty, p^\infty$ [we can, for example, take the curl of Eq. (24) to eliminate ∇p^∞ and solve for $\boldsymbol{u}^\infty$, together with taking the divergence of Eq. (24) and solve the Poisson equation for p^∞]. From the solution of Eqs. (23) and (24), we can solve for ρ^∞ from Eq. (25). These equations were derived by eliminating of all fast-time scale variation and assuming that the solar wind background is in an equilibrium, nonevolutionary state. To eliminate fast-time variation, we of course have to prescribe initial data which do not contain fast fluctuations. Small-scale fluctuations and solar wind background components are coupled through the parameter χ, which is the ratio of typical lengthscales for small-scale fluctuations (L) and the large-scale solar wind background (R), $\chi = \frac{L}{R}$. In the case that large-scale gradients are not present in the solar wind, χ can vanish, and as the limit in this special case we obtain usual hydrodynamic incompressible equations. In summary, the locally incompressible model (23)–(25) shows the following.

Large-scale gradients in the solar wind density (pressure) introduce a nontrivial component of nonsolenoidal velocity fluctuations.

The nonsolenoidal velocity fluctuations generate significant density perturbations at an order $O(M_{s0})$. These density fluctuations are thus highly subsonic, convected structures that will be created even if they are absent in the initial data, which stands in contrast with homogeneous flow.

Locally incompressible velocity, density, and pressure fields are coupled to and driven by large-scale gradients in the solar wind, which leads to a rich and dynamically complex evolution of solar wind variables and can have important implications for understanding solar wind turbulence.

For deeper insight into the equations, it is useful to introduce convective (slow) and acoustic (fast) times scales

$$\tau = t, \quad \tau' = \frac{t}{\epsilon} \Rightarrow \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \frac{1}{\epsilon}\frac{\partial}{\partial \tau'}, \tag{26}$$

and short, respectively, long wavelengths

$$\eta = \boldsymbol{x}, \quad \xi = \epsilon\mathbf{x} \Rightarrow \nabla = \nabla_\eta + \epsilon\nabla_\xi, \tag{27}$$

and apply these operators to the LI equations (23)–(25). We expand the nabla operators that apply to the fluctuations only and not to the solar wind variables. To emphasize the distinction, we introduce the superscript "r" to the nabla operator that applies to the background solar wind, to remind us that it represents $\frac{\partial}{\partial r}$ in the radial direction. We should also note, that henceforth, we take χ to be of order ϵ.

From Eq. (23), we obtain

$$O(\epsilon^0):\ \nabla_\eta \cdot \boldsymbol{u}^\infty = 0, \tag{28}$$

$$O(\epsilon^1):\ \nabla_\xi \cdot \boldsymbol{u}^\infty = -\frac{\chi}{\epsilon\rho_{\rm SW}}\boldsymbol{u}^\infty\cdot\nabla_r\rho_{\rm SW}, \tag{29}$$

from Eq. (24)

$$O\left(\frac{1}{\epsilon}\right):\ \frac{\partial \boldsymbol{u}^\infty}{\partial \tau'} = 0, \tag{30}$$

$$O(\epsilon^0):\ \rho_{\rm SW}\frac{\partial \boldsymbol{u}^\infty}{\partial \tau} + \rho_{\rm SW}(\boldsymbol{u}_{\rm SW}+\boldsymbol{u}^\infty)\cdot\nabla_\eta\boldsymbol{u}^\infty = -\nabla_\eta p^\infty; \tag{31}$$

$$O(\epsilon^1):\ \rho_{\rm SW}(\boldsymbol{u}_{\rm SW}+\boldsymbol{u}^\infty)\,.\,\nabla_\xi\boldsymbol{u}^\infty = -\nabla_\xi p^\infty - \frac{\chi}{\epsilon}\rho_{\rm SW}\boldsymbol{u}^\infty\cdot\nabla_r\boldsymbol{u}_{\rm SW}, \tag{32}$$

and from Eqs. (25) or (21)

$$O\left(\frac{1}{\epsilon}\right):\ \frac{\partial \rho^\infty}{\partial \tau'} = 0, \tag{33}$$

$$O(\epsilon^0):\ \frac{\partial \rho^\infty}{\partial \tau} + (\boldsymbol{u}_{\rm SW}+\boldsymbol{u}^\infty)\cdot\nabla_\eta\rho^\infty = 0, \tag{34}$$

$$O(\epsilon^1):\ (\boldsymbol{u}_{\rm SW}+\boldsymbol{u}^\infty)\cdot\nabla_\xi\rho^\infty + \rho^\infty\nabla_\xi\cdot\boldsymbol{u}^\infty = -\frac{\chi}{\epsilon}\rho^\infty\nabla_r\cdot\boldsymbol{u}_{\rm SW}. \tag{35}$$

In the last equation it is better to retain the term with $\nabla_\xi \cdot \boldsymbol{u}^\infty$ rather than use Eq. (29), because these equations will be used to cancel terms in the derivation of the nearly incompressible equations below. It is interesting to note that Eq. (28) and (31), together with Eq. (34), [all of them of order $O(\epsilon^0)$] are consistent with the phenomenology of a leading order incompressible flow, now written for short length scales. This is the reason why the more appropriate term "locally incompressible equations" was chosen.

Similarly, high frequency motions at this order are also suppressed [see Eq. (30)]. The locally incompressible fluid motion is thus devoid of high-frequency fluctuations. Equations (28)–(35) illustrate explicitly that the LI equations (23)–(25) express variations in the fluctuations on slow (convective) time scales and are independent of acoustic variations. The multiple spatial-scale expansion also illustrates that it is the long-wavelength, low-frequency fluctuations that couple to and are driven by the large-scale (solar wind) gradient. In the following section, we develop a hierarchy of fluid equations that possess disparate length and time scales, and that provide a higher-order correction to the core LI solutions in the framework of a nearly incompressible theory.

V. NEARLY INCOMPRESSIBLE CORRECTIONS

Consider now the fully compressible equations (10), (12), and (13) and introduce the nearly incompressible (NI) corrections $\boldsymbol{u}_1, p^*$ and ρ_2 to the locally incompressible (LI) low frequency variables $\boldsymbol{u}^\infty, p^\infty$, and ρ^∞ according to the expansion

$$\tilde{\mathbf{u}}' = \boldsymbol{u}^\infty + \epsilon \boldsymbol{u}_1,$$

$$\tilde{p}' = \epsilon^2 (p^\infty + p^*),$$

$$\tilde{\rho}' = \epsilon \rho^\infty + \epsilon^2 \rho_2. \tag{36}$$

Note, that in contrast with NI velocity and density corrections, the NI pressure p^* is introduced at the same order as the pressure p^∞. This is motivated by the earlier homogeneous work of Zank and Matthaeus [5,6], where it was shown that to obtain correct nearly incompressible equations, it is necessary to perform the NI expansion in the fashion of Eq. (36). This can be justified more formally from the rigorous work of Klainerman and Majda [2] (for example, theorem 2, section "Justification of linearized acoustics," together with Sec. III, "The Proof of Theorem 2"), where they explicitly consider (in their slightly different notation) the NI pressure p^* to be at the same ϵ^2 order as pressure p^∞. The justification for introducing p^* at $O(\epsilon^2)$ arises from the presence of the second-order density fluctuation. Its fast-scale presence has to be balanced by a corresponding fast-scale response in pressure otherwise the only consistent solution would be $\rho_2 = 0$, and hence no acoustic fluctuations. Thus, to include acoustic (or equivalently compressible) variations in the next order we have to introduce a compressible $O(\epsilon^2)$ pressure term.

Since, typically in the solar wind, $u_0 \ll c_{s0}$ and $\epsilon = \gamma^{1/2} M_{s0} \ll 1$, the expansion is convergent and does not create nonphysically growing solutions. In the situation when $M_{s0} \gg 1$, one has to use a fully compressible fluid model that accounts for both subsonic and supersonic fluid motion, but this is not of interest here.

The locally incompressible variables explicitly satisfy Eqs. (23)–(25). We adopt the same approach used by Zank and Matthaeus and include the time and wavelength scale separation (26) and (27). Consider the normalized momentum equation (12), which, after using the expansion (36), yields at the first order

$$O\left(\frac{1}{\epsilon}\right): \ \rho_{\mathrm{SW}} \frac{\partial \boldsymbol{u}^\infty}{\partial \tau'} = 0,$$

which was already obtained in Eq. (30). At the next order,

$$O(\epsilon^0): \ \rho_{\mathrm{SW}} \frac{\partial \boldsymbol{u}^\infty}{\partial \tau} + \rho^\infty \frac{\partial \boldsymbol{u}^\infty}{\partial \tau'} + \rho_{\mathrm{SW}} \frac{\partial \boldsymbol{u}_1}{\partial \tau'} + \rho_{\mathrm{SW}} (\boldsymbol{u}_{\mathrm{SW}} + \boldsymbol{u}^\infty) \cdot \nabla_\eta \boldsymbol{u}^\infty = -\nabla_\eta (p^\infty + p^*).$$

After cancellation of terms due to the LI equations (30) and (31), we obtain

$$O(\epsilon^0): \ \rho_{\mathrm{SW}} \frac{\partial \boldsymbol{u}_1}{\partial \tau'} = -\nabla_\eta p^*. \tag{37}$$

This is the same equation derived by Zank and Matthaeus [their Eq. (32)] and evidently, p^* is the pressure associated with nonconvective fluid motions, and Eq. (37) indicates that short wavelengths are associated with high-frequency acoustic fluctuations. Continuing to the next order and using Eqs. (30) and (32) yields

$$O(\epsilon^1): \ \rho_{\mathrm{SW}} \frac{\partial \boldsymbol{u}_1}{\partial \tau} + \rho^\infty \frac{\partial \boldsymbol{u}_1}{\partial \tau'} + \rho^\infty \frac{\partial \boldsymbol{u}^\infty}{\partial \tau} + \rho_{\mathrm{SW}} (\boldsymbol{u}_{\mathrm{SW}} + \boldsymbol{u}^\infty) \cdot \nabla_\eta \boldsymbol{u}_1 + \rho^\infty (\boldsymbol{u}_{\mathrm{SW}} + \boldsymbol{u}^\infty) \cdot \nabla_\eta \boldsymbol{u}^\infty + \rho_{\mathrm{SW}} \boldsymbol{u}_1 \cdot \nabla_\eta \boldsymbol{u}^\infty = -\nabla_\xi p^*, \tag{38}$$

which represents the "linearized" momentum equation. The linearization is in terms of the highly nonlinear background variables $\boldsymbol{u}^\infty, \rho^\infty$, which are present both as coefficients and source terms, as are the background prescribed variables $\rho_{\mathrm{SW}}, p_{\mathrm{SW}}$. Equation (38) is therefore very complicated to solve, but some insight can be gained into its basic structure by considering the simplification $\boldsymbol{u}^\infty = 0$. In this case we obtain

$$\rho_{\mathrm{SW}} \frac{\partial \boldsymbol{u}_1}{\partial \tau} + \rho^\infty \frac{\partial \boldsymbol{u}_1}{\partial \tau'} + \rho_{\mathrm{SW}} \boldsymbol{u}_{\mathrm{SW}} \cdot \nabla_\eta \boldsymbol{u}_1 = -\nabla_\xi p^*, \tag{39}$$

which is the long wavelength analog of Eq. (37) modified by additional solar wind terms. On continuing the expansion to the next order, we obtain

$$O(\epsilon^2): \quad \rho^\infty \frac{\partial \boldsymbol{u}_1}{\partial \tau} + \rho_2 \frac{\partial \boldsymbol{u}^\infty}{\partial \tau} + \rho_2 \frac{\partial \boldsymbol{u}_1}{\partial \tau'} + \rho_{\mathrm{SW}} (\boldsymbol{u}_{\mathrm{SW}} + \boldsymbol{u}^\infty) \cdot \nabla_\xi \boldsymbol{u}_1 + \rho_{\mathrm{SW}} \boldsymbol{u}_1 \cdot (\nabla_\eta \boldsymbol{u}_1 + \nabla_\xi \boldsymbol{u}^\infty) + \rho^\infty (\boldsymbol{u}_{\mathrm{SW}} + \boldsymbol{u}^\infty) \cdot (\nabla_\eta \boldsymbol{u}_1 + \nabla_\xi \boldsymbol{u}^\infty) + \rho^\infty \boldsymbol{u}_1 \cdot \nabla_\eta \boldsymbol{u}^\infty + \rho_2 (\boldsymbol{u}_{\mathrm{SW}} + \boldsymbol{u}^\infty) \cdot \nabla_\eta \boldsymbol{u}^\infty = -\frac{\chi}{\epsilon} \rho^\infty (\boldsymbol{u}^\infty + \boldsymbol{u}_{\mathrm{SW}}) \cdot \nabla_r \boldsymbol{u}_{\mathrm{SW}} - \frac{\chi}{\epsilon} \rho_{\mathrm{SW}} \boldsymbol{u}_1 \cdot \nabla_r \boldsymbol{u}_{\mathrm{SW}}. \tag{40}$$

For completeness we write down one more order:

$$O(\epsilon^3):\ \rho_2\frac{\partial \boldsymbol{u}_1}{\partial \tau} + \rho^\infty \boldsymbol{u}_{\rm SW}\cdot\nabla_\xi \boldsymbol{u}_1 + \rho_2 \boldsymbol{u}_{\rm SW}\cdot(\nabla_\xi \boldsymbol{u}^\infty + \nabla_\eta \boldsymbol{u}_1)$$
$$+ \rho_{\rm SW}\boldsymbol{u}_1\cdot\nabla_\xi \boldsymbol{u}_1 + \rho^\infty \boldsymbol{u}^\infty\cdot\nabla_\xi \boldsymbol{u}_1 + \rho^\infty \boldsymbol{u}_1\cdot(\nabla_\eta \boldsymbol{u}_1 + \nabla_\xi \boldsymbol{u}^\infty)$$
$$+ \rho_2 \boldsymbol{u}^\infty\cdot(\nabla_\eta \boldsymbol{u}_1 + \nabla_\xi \boldsymbol{u}^\infty) + \rho_2 \boldsymbol{u}_1\cdot\nabla_\eta \boldsymbol{u}^\infty$$
$$= -\frac{\chi}{\epsilon}\rho^\infty \boldsymbol{u}_1\cdot\nabla_r \boldsymbol{u}_{\rm SW} - \frac{\chi}{\epsilon}\rho_2(\boldsymbol{u}_{\rm SW}+\boldsymbol{u}^\infty)\cdot\nabla_r \boldsymbol{u}_{\rm SW}. \tag{41}$$

Equations (37), (38), (40), and (41), all in different orders of ϵ, can be combined and represented as

$$\rho_{\rm SW}\frac{\partial \boldsymbol{u}_1}{\partial t} + (\rho^\infty + \epsilon\rho_2)\frac{\partial}{\partial t}(\boldsymbol{u}^\infty + \epsilon \boldsymbol{u}_1) + \rho_{\rm SW}(\boldsymbol{u}_{\rm SW}+\boldsymbol{u}^\infty)\cdot\nabla \boldsymbol{u}_1$$
$$+ (\rho^\infty + \epsilon\rho_2)\boldsymbol{u}_{\rm SW}\cdot\nabla(\boldsymbol{u}^\infty + \epsilon \boldsymbol{u}_1) + \rho_{\rm SW}\boldsymbol{u}_1\cdot\nabla(\boldsymbol{u}^\infty + \epsilon \boldsymbol{u}_1)$$
$$+ (\rho^\infty + \epsilon\rho_2)(\boldsymbol{u}^\infty + \epsilon \boldsymbol{u}_1)\cdot\nabla(\boldsymbol{u}^\infty + \epsilon \boldsymbol{u}_1)$$
$$= -\frac{\nabla p^*}{\epsilon} - \chi(\rho^\infty + \epsilon\rho_2)\boldsymbol{u}_{\rm SW}\cdot\nabla_r \boldsymbol{u}_{\rm SW} - \chi\rho_{\rm SW}\boldsymbol{u}_1\cdot\nabla_r \boldsymbol{u}_{\rm SW}$$
$$- \chi(\rho^\infty + \epsilon\rho_2)(\boldsymbol{u}^\infty + \epsilon \boldsymbol{u}_1)\cdot\nabla_r \boldsymbol{u}_{\rm SW}. \tag{42}$$

At this point, it is worth commenting that reconstructing the multiple-scales expanded equations (37)–(41) to obtain Eq. (42) is a tedious procedure. Although the separation of variable into long/short wavelengths and slow/fast time scales provides insight into the various coupling and source terms, and clarifies the nature of the NI expansion, this procedure is not necessary for deriving the final nearly incompressible model. One can instead expand the compressible equations using the NI expansion (36) and the locally incompressible equations (23)–(25) to directly obtain Eq. (42). Keeping just the lowest order terms in Eq. (42) yields the inhomogeneous nearly incompressible momentum equation

$$\rho_{\rm SW}\frac{\partial \boldsymbol{u}_1}{\partial t} + \rho^\infty\frac{\partial \boldsymbol{u}^\infty}{\partial t} + \rho_{\rm SW}(\boldsymbol{u}_{\rm SW}+\boldsymbol{u}^\infty)\cdot\nabla \boldsymbol{u}_1$$
$$+ \rho^\infty(\boldsymbol{u}_{\rm SW}+\boldsymbol{u}^\infty)\cdot\nabla \boldsymbol{u}^\infty + \rho_{\rm SW}\boldsymbol{u}_1\cdot\nabla \boldsymbol{u}^\infty$$
$$= -\frac{1}{\epsilon}\nabla p^*. \tag{43}$$

This equation is analogous to that derived by Zank and Matthaeus [their Eq. 36] and it shows how the solar wind and LI velocity field is coupled to the "acoustic" velocity field, demonstrating how momentum is transfered from the solar wind and locally incompressible field to the acoustic field. This can be considered as a generalization of the Lighthill method for the generation of sound (Lighthill [30]).

Consider now the relationship between Eq. (43) in the limit of no solar wind, and that published by ZM. For that, we have to compare the expansions used here and in ZM:

$$\text{Now: } \tilde{\boldsymbol{u}} = \boldsymbol{u}_{\rm SW} + \boldsymbol{u}^\infty + \epsilon \boldsymbol{u}_1, \quad \text{ZM: } \tilde{\boldsymbol{u}} = \boldsymbol{u}^\infty + \epsilon \boldsymbol{u}_1,$$
$$\tilde{p} = p_{\rm SW} + \epsilon^2(p^\infty + p^*), \qquad \tilde{p} = 1 + \epsilon^2(p^\infty + p^*),$$
$$\tilde{\rho} = \rho_{\rm SW} + \epsilon\rho^\infty + \epsilon^2\rho_2, \qquad \tilde{\rho} = 1 + \epsilon^2\rho_1.$$

Evidently, from the expansions, the correct limit requires that

$$\boldsymbol{u}_{\rm SW} \to \boldsymbol{0},$$
$$\rho_{\rm SW} \to 1, \quad \rho^\infty \to \boldsymbol{0}.$$
$$p_{\rm SW} \to 1. \tag{44}$$

Using this limit, Eq. (43) reduces to

$$\frac{\partial \boldsymbol{u}_1}{\partial t} + \boldsymbol{u}^\infty\cdot\nabla \boldsymbol{u}_1 + \boldsymbol{u}_1\cdot\nabla \boldsymbol{u}^\infty = -\frac{1}{\epsilon}\nabla p^*,$$

which is exactly the nearly incompressible momentum equation published by Zank and Matthaeus (in the absence of viscous terms).

On using the limit (44), we can also demonstrate that the locally incompressible equations (23) and (24), reduce to the usual incompressible equations

$$\nabla\cdot \boldsymbol{u}^\infty = 0,$$
$$\frac{\partial \boldsymbol{u}^\infty}{\partial t} + \boldsymbol{u}^\infty\cdot\nabla \boldsymbol{u}^\infty = -\nabla p^\infty,$$

together with (25), which reduces to $D\rho^\infty/Dt=0$.

Consider now the energy equation (13), and let us reverse the procedure developed above and first derive the NI equations before introducing multiple scales. Doing the nearly incompressible expansion (36) of the energy equation (13) yields

$$\epsilon^2\frac{\partial}{\partial t}(p^\infty + p^*) + \epsilon^2(\boldsymbol{u}_{\rm SW}+\boldsymbol{u}^\infty)\cdot\nabla(p^\infty + p^*) + \epsilon^3\boldsymbol{u}_1\cdot\nabla(p^\infty + p^*)$$
$$+ \gamma p_{\rm SW}\nabla\cdot \boldsymbol{u}^\infty + \epsilon^2\gamma(p^\infty + p^*)\nabla\cdot \boldsymbol{u}^\infty + \epsilon\gamma p_{\rm SW}\nabla\cdot \boldsymbol{u}_1$$
$$+ \epsilon^3\gamma(p^\infty + p^*)\nabla\cdot \boldsymbol{u}_1$$
$$= -\chi \boldsymbol{u}^\infty\cdot\nabla_r p_{\rm SW} - \chi\epsilon \boldsymbol{u}_1\cdot\nabla_r p_{\rm SW}$$
$$- \gamma\chi\epsilon^2(p^\infty + p^*)\nabla_r\cdot \boldsymbol{u}_{\rm SW}.$$

After using the LI equation (20), dividing by ϵ^2, and taking just the lowest order terms in ϵ [remember $\nabla\cdot\mathbf{u}^\infty \sim O(\chi)$] gives

$$\frac{\partial}{\partial t}(p^\infty + p^*) + (\boldsymbol{u}_{\rm SW}+\boldsymbol{u}^\infty)\cdot\nabla(p^\infty + p^*) + \frac{1}{\epsilon}\gamma p_{\rm SW}\nabla\cdot \boldsymbol{u}_1$$
$$= -\frac{\chi}{\epsilon}\boldsymbol{u}_1\cdot\nabla_r p_{\rm SW}. \tag{45}$$

This is the inhomogeneous nearly incompressible energy equation in the presence of a large-scale solar wind background. In the absence of the background solar wind, the nearly incompressible equation (45), using the limit (44), reduces to

$$\frac{\partial}{\partial t}(p^\infty + p^*) + \boldsymbol{u}^\infty\cdot\nabla(p^\infty + p^*) + \frac{1}{\epsilon}\gamma\nabla\cdot \boldsymbol{u}_1 = 0, \tag{46}$$

which is the same equation given by Zank and Matthaeus [ZM, their Eq. (44)].

Consider now the corresponding multiple-scale expansion of the energy equation. We obtain at the first order

$$O(\epsilon^0):\ \gamma p_{\mathrm{SW}} \boldsymbol{\nabla}_{\eta} \cdot \boldsymbol{u}^{\infty} = 0,$$

which is in agreement with Eq. (28). Continuing to higher order

$$O(\epsilon^1):\ \frac{\partial}{\partial \tau'}(p^{\infty} + p^*) + \gamma p_{\mathrm{SW}}(\boldsymbol{\nabla}_{\xi} \cdot \boldsymbol{u}^{\infty} + \boldsymbol{\nabla}_{\eta} \cdot \boldsymbol{u}_1)$$
$$= - \frac{\chi}{\epsilon}\boldsymbol{u}^{\infty} \cdot \boldsymbol{\nabla}_r p_{\mathrm{SW}},$$

and recalling that p^{∞} is independent of the fast time scale τ', together with suitable cancellations (28) and (29), leads to

$$O(\epsilon^1):\ \frac{\partial p^*}{\partial \tau'} + \gamma p_{\mathrm{SW}} \boldsymbol{\nabla}_{\eta} \cdot \boldsymbol{u}_1 = 0. \tag{47}$$

This equation is analogous to ZM equation [their Eq. (37)]. After combining Eqs. (37) and (47), recalling that the solar wind variables are time independent and making the reasonable assumption that they do not depend also on short wavelengths ($\boldsymbol{\nabla}_{\eta} p_{\mathrm{SW}}=0, \boldsymbol{\nabla}_{\eta}\rho_{\mathrm{SW}}=0$), we obtain

$$\frac{\partial^2 p^*}{\partial \tau'^2} - \frac{\gamma p_{\mathrm{SW}}}{\rho_{\mathrm{SW}}}\boldsymbol{\nabla}^2_{\eta} p^* = 0,$$
$$\frac{\partial^2 \boldsymbol{u}_1}{\partial \tau'^2} - \frac{\gamma p_{\mathrm{SW}}}{\rho_{\mathrm{SW}}}\boldsymbol{\nabla}^2_{\eta} \boldsymbol{u}_1 = 0. \tag{48}$$

These equations correspond to wave equations for the pressure and velocity perturbations p^* and $\boldsymbol{u}_1$ on the fast time/short wavelength scales. Because $\frac{\gamma p_{\mathrm{SW}}}{\rho_{\mathrm{SW}}}=c_s^2$, these waves or perturbations propagate at the sound speed defined in terms of the inhomogeneous background state $(p_{\mathrm{SW}},\rho_{\mathrm{SW}})$. However, since the background sound speed is independent of the fast time, short wavelength variables τ', η, Eqs. (48) remain nonetheless homogeneous wave equations, at this order. On the other hand, the sound speed c_s is spatially inhomogeneous and governed by the large-scale solar wind equilibrium. In the analogous ZM equations, the normalized pressure and density is proportional to one. The speed of sound in their equations is therefore "hidden" and constant. For completeness, we note that even though the wave equations (48) are sourceless, there is no reason to exclude fluctuations in p^* and $\boldsymbol{u}_1$ from initial data, since they produce only bounded acceleration as $\epsilon \to 0$ (which is in contrast with p_1 as was discussed in Sec. IV), and agrees with Matthaeus and Brown [3]. Continuing the expansion of Eq. (13) to higher orders and using Eq. (28) gives

$$O(\epsilon^2):\ \frac{\partial}{\partial \tau}(p^{\infty} + p^*) + (\boldsymbol{u}_{\mathrm{SW}} + \boldsymbol{u}^{\infty}) \cdot \boldsymbol{\nabla}_{\eta}(p^{\infty} + p^*)$$
$$+ \gamma p_{\mathrm{SW}} \boldsymbol{\nabla}_{\xi} \cdot \boldsymbol{u}_1 = - \frac{\chi}{\epsilon}\boldsymbol{u}_1 \cdot \boldsymbol{\nabla}_r p_{\mathrm{SW}}. \tag{49}$$

This equation describes the evolution of acoustic energy on slow time scales, and individual terms illustrate how the acoustic pressure p^* is driven by short and long wavelength velocity fluctuations and the background solar wind. Exploring the special case when $\boldsymbol{u}^{\infty}=0$, which implies that p^{∞} can at best be constant, reduces Eq. (49) to

$$\frac{\partial p^*}{\partial \tau} + \boldsymbol{u}_{\mathrm{SW}} \cdot \boldsymbol{\nabla}_{\eta} p^* + \gamma p_{\mathrm{SW}} \boldsymbol{\nabla}_{\xi} \cdot \boldsymbol{u}_1 = - \frac{\chi}{\epsilon}\boldsymbol{u}_1 \cdot \boldsymbol{\nabla}_r p_{\mathrm{SW}}.$$

One can in principle couple this equation to Eqs. (39) and (47) to obtain slow time/long wavelength wave equations for quantities p^* and $\boldsymbol{u}_1$ [similar to ZM, their Eq. (40)], but because of the complicated solar wind terms, this does not yield anything tractable. The higher-order expansion is given by

$$O(\epsilon^3):\ (\boldsymbol{u}_{\mathrm{SW}} + \boldsymbol{u}^{\infty}) \cdot \boldsymbol{\nabla}_{\xi}(p^{\infty} + p^*) + \boldsymbol{u}_1 \cdot \boldsymbol{\nabla}_{\eta}(p^{\infty} + p^*) + \gamma(p^{\infty}$$
$$+ p^*)(\boldsymbol{\nabla}_{\eta} \cdot \boldsymbol{u}_1 + \boldsymbol{\nabla}_{\xi} \cdot \boldsymbol{u}^{\infty}) = - \gamma\frac{\chi}{\epsilon}(p^{\infty} + p^*)\boldsymbol{\nabla}_r \cdot \boldsymbol{u}_{\mathrm{SW}}. \tag{50}$$

Coupling equations (47), (49), and (50), one can, at the lowest order, obtain again the nearly incompressible energy equation (45).

Finally, we can derive the nearly incompressible continuity equation. From Eq. (10), we find

$$\epsilon\frac{\partial \rho^{\infty}}{\partial t} + \epsilon^2\frac{\partial \rho_2}{\partial t} + \epsilon\rho^{\infty}\boldsymbol{\nabla} \cdot \boldsymbol{u}^{\infty} + \epsilon^2\rho_2\boldsymbol{\nabla} \cdot \boldsymbol{u}^{\infty} + \epsilon^2(\rho^{\infty}$$
$$+ \epsilon\rho_2)\boldsymbol{\nabla} \cdot \boldsymbol{u}_1 + \epsilon(\boldsymbol{u}_{\mathrm{SW}} + \boldsymbol{u}^{\infty}) \cdot \boldsymbol{\nabla}\rho^{\infty} + \epsilon^2\boldsymbol{u}_1 \cdot \boldsymbol{\nabla}\rho^{\infty} + \epsilon^2(\boldsymbol{u}_{\mathrm{SW}}$$
$$+ \boldsymbol{u}^{\infty}) \cdot \boldsymbol{\nabla}\rho_2 + \epsilon^3\boldsymbol{u}_1 \cdot \boldsymbol{\nabla}\rho_2 + \rho_{\mathrm{SW}}\boldsymbol{\nabla} \cdot \boldsymbol{u}^{\infty} + \epsilon\rho_{\mathrm{SW}}\boldsymbol{\nabla} \cdot \boldsymbol{u}_1$$
$$= - \chi\epsilon\rho^{\infty}\boldsymbol{\nabla}_r \cdot \boldsymbol{u}_{\mathrm{SW}} - \chi\epsilon^2\rho_2\boldsymbol{\nabla}_r \cdot \boldsymbol{u}_{\mathrm{SW}} - \chi\boldsymbol{u}^{\infty} \cdot \boldsymbol{\nabla}_r\rho_{\mathrm{SW}}$$
$$- \chi\epsilon\boldsymbol{u}_1 \cdot \boldsymbol{\nabla}_r\rho_{\mathrm{SW}}.$$

As before, using locally incompressible equations (23) and (25), and dividing by ϵ^2 yields

$$\frac{\partial \rho_2}{\partial t} + (\boldsymbol{u}_{\mathrm{SW}} + \boldsymbol{u}^{\infty}) \cdot \boldsymbol{\nabla}\rho_2 + \rho_2\boldsymbol{\nabla} \cdot \boldsymbol{u}^{\infty} + \rho^{\infty}\boldsymbol{\nabla} \cdot \boldsymbol{u}_1 + \epsilon\rho_2\boldsymbol{\nabla} \cdot \boldsymbol{u}_1$$
$$+ \boldsymbol{u}_1 \cdot \boldsymbol{\nabla}\rho^{\infty} + \epsilon\boldsymbol{u}_1 \cdot \boldsymbol{\nabla}\rho_2 + \frac{1}{\epsilon}\rho_{\mathrm{SW}}\boldsymbol{\nabla} \cdot \boldsymbol{u}_1$$
$$= - \frac{\chi}{\epsilon}\boldsymbol{u}_1 \cdot \boldsymbol{\nabla}_r\rho_{\mathrm{SW}} - \chi\rho_2\boldsymbol{\nabla}_r \cdot \boldsymbol{u}_{\mathrm{SW}}. \tag{51}$$

Retaining the lowest order terms, we obtain the inhomogeneous nearly incompressible continuity equation, including the solar wind background, as

$$\frac{\partial \rho_2}{\partial t} + \rho^{\infty}\boldsymbol{\nabla} \cdot \boldsymbol{u}_1 + \boldsymbol{u}_1 \cdot \boldsymbol{\nabla}\rho^{\infty} + (\boldsymbol{u}_{\mathrm{SW}} + \boldsymbol{u}^{\infty}) \cdot \boldsymbol{\nabla}\rho_2$$
$$= - \frac{1}{\epsilon}\rho_{\mathrm{SW}}\boldsymbol{\nabla} \cdot \boldsymbol{u}_1 - \frac{\chi}{\epsilon}\boldsymbol{u}_1 \cdot \boldsymbol{\nabla}_r\rho_{\mathrm{SW}}. \tag{52}$$

We may verify that the NI continuity equation (52) reduces, in the no solar wind limit (44), to

$$\frac{\partial \rho_2}{\partial t} + \boldsymbol{u}^{\infty} \cdot \boldsymbol{\nabla}\rho_2 + \frac{1}{\epsilon}\boldsymbol{\nabla} \cdot \boldsymbol{u}_1 = 0,$$

which is exactly the nearly incompressible continuity equation published in Zank and Matthaeus [their Eq. 51].

We may proceed as before using multiple scales time and wavelength expansion (26) and (27), together with NI expansion (36) for the continuity equation (13). At the first order, we find

$$O(\epsilon^0):\ \frac{\partial \rho^\infty}{\partial \tau'} + \rho_{SW}\nabla_\eta \cdot \boldsymbol{u}^\infty = 0,$$

and both terms independently go to zero. A further expansion yields

$$O(\epsilon^1):\quad \frac{\partial \rho_2}{\partial \tau'} + \rho_{SW}\nabla_\eta \cdot \boldsymbol{u}_1 = 0, \tag{53}$$

while at higher orders, we have

$$O(\epsilon^2):\quad \frac{\partial \rho_2}{\partial \tau} + (\boldsymbol{u}_{SW} + \boldsymbol{u}^\infty)\cdot \nabla_\eta \rho_2 + \rho^\infty \nabla_\eta \cdot \boldsymbol{u}_1 + \boldsymbol{u}_1 \cdot \nabla_\eta \rho^\infty + \rho_{SW}\nabla_\xi \cdot \boldsymbol{u}_1 = -\frac{\chi}{\epsilon}\boldsymbol{u}_1 \cdot \nabla_r \rho_{SW}, \tag{54}$$

$$O(\epsilon^3):\quad \boldsymbol{u}_{SW}\cdot \nabla_\xi \rho_2 + \rho^\infty \nabla_\xi \cdot \boldsymbol{u}_1 + \rho_2 \nabla_\eta \cdot \boldsymbol{u}_1 + \rho_2 \nabla_\xi \cdot \boldsymbol{u}^\infty + \boldsymbol{u}_1 \cdot \nabla_\xi \rho^\infty + \boldsymbol{u}_1 \cdot \nabla_\eta \rho_2 + \boldsymbol{u}^\infty \cdot \nabla_\xi \rho_2 = -\frac{\chi}{\epsilon}\rho_2 \nabla_r \cdot \boldsymbol{u}_{SW}. \tag{55}$$

On reconstituting Eqs. (53)–(55), we recover at the lowest order, the nearly incompressible equation of continuity Eq. (52).

VI. GENERALIZED PSEUDOSOUND RELATION

In an adiabatically compressive fluid, the pressure and density variations can be related through the sound speed. Such a relationship, in our present analysis, can be deduced readily by combining Eqs. (53) and (47) as follows,

$$\frac{\partial p^*}{\partial \tau'} = \gamma \frac{p_{SW}}{\rho_{SW}} \frac{\partial \rho_2}{\partial \tau'}. \tag{56}$$

The above equation is slightly different from the one obtained in ZM [their Eq. (48)] for an isotropic and homogeneous fluid, and because solar wind quantities are time independent, it does conveniently lead to a linear relationship between the fluctuations ρ_2 and p^*

$$p^* = c_s(r)^2 \rho_2, \quad \text{where } c_s(r)^2 = \gamma \frac{p_{SW}}{\rho_{SW}},$$

implying that the fluctuations propagate with the speed of sound. This is valid both for normalized or non-normalized quantities p^* and ρ_2 as the solar wind p_{SW} and ρ_{SW} are normalized with respect to the same p_0,ρ_0.

This is the basis of the pseudosound approximation, used to relate density and pressure fluctuations. The sound speed is, however, in fact determined by the large-scale motion. This is fundamentally different from ZM inference that c_s was constant in the wave equation and determined by the normalizing quantities. Because the locally incompressible pressure p^∞ does not depend on the fast-time scales τ', we can add the pressure p^∞ under $\partial/\partial\tau'$ in Eq. (56) to obtain a generalized relation for fast-time/short-wavelength scales

$$p^\infty + p^* = c_s(r)^2 \rho_2. \tag{57}$$

It is to be born in mind that Eq. (57) is slightly different for this inhomogeneous case however, in that it possesses the sound speed $c_s(r)^2$ instead of c_{s0}^2. The latter, as obtained by ZM, is determined by the normalizing p_0, ρ_0 and is therefore held constant, whereas the former (i.e., c_s^2) depends upon the local large-scale solar wind background density and pressure gradients. The inhomogeneous sound speed thus varies locally and its dependence can be estimated as follows. Let us assume that $\boldsymbol{u}_{SW}$ is a constant and is described by a mean flow speed. It then follows from the continuity equation that the density scales as $\rho_{SW} \sim 1/r^2$. The solar wind, being an adiabatic fluid, obeys the adiabatic $p_{SW} = \rho_{SW}^\gamma$ relationship. This gives us $p_{SW} \sim 1/r^{2\gamma}$ and finally $c_s(r) \sim 1/r^{\gamma-1}$.

To derive a similar relation for the slow-time/long-wavelength scales is more complicated. On using a similar construction as in ZM [5], the term proportional to $\nabla_\xi \cdot \boldsymbol{u}_1$ in the acoustic energy equation (49) is eliminated by using Eqs. (54) and (14), and rewritten as

$$\frac{\partial}{\partial \tau}(p^\infty + p^*) + (\boldsymbol{u}_{SW} + \boldsymbol{u}^\infty)\cdot \nabla_\eta (p^\infty + p^*) - c_s^2 \frac{\partial \rho_2}{\partial \tau} - c_s^2 (\boldsymbol{u}_{SW} + \boldsymbol{u}^\infty)\cdot \nabla_\eta \rho_2 - c_s^2 \rho^\infty \nabla_\eta \cdot \boldsymbol{u}_1 - c_s^2 \boldsymbol{u}_1 \cdot \nabla_\eta \rho^\infty = 0, \tag{58}$$

This is too complicated to deduce a simple form such as Eq. (57). The coupling between the large-scale solar wind nonevolutionary equilibrium and the turbulent fluctuations is primarily responsible for the complexities of Eq. (58). However, two cases allow for a simple analytic understanding of Eq. (58).

A. Case I: No solar wind background

Let us assume that there exists no large-scale solar wind background. According to the limit Eq. (44), we can eliminate ρ^∞ and Eq. (58) then yields

$$\frac{\partial}{\partial \tau}(p^\infty + p^*) + \boldsymbol{u}^\infty \cdot \nabla_\eta (p^\infty + p^*) - c_{s0}^2 \frac{\partial \rho_2}{\partial \tau} - c_{s0}^2 \boldsymbol{u}^\infty \cdot \nabla_\eta \rho_2 = 0, \tag{59}$$

which converges to the expression $p^\infty + p^* = c_{s0}^2 \rho_2$, which is identical to that obtained in ZM in the limit $c_s(r) \to c_{s0}$. However, when the solar wind background is present, it is analytically intractable to deduce a generalized pseudosound relation in the presence of large-scale SW background source terms. For this reason, a simple pseudosound relation cannot be obtained and Eq. (58) represents an implicit relationship between p^∞, p^*, and ρ_2.

B. Case II: Background solar wind pressure

In this case we continue to work with full Eq. (58) without any specific assumptions. As can be seen, the linear pseudosound relationship, $p^{\infty}+p^{*}=c_s^2\rho_2$, still satisfies the homogeneous part (59) of inhomogeneous Eq. (58). To obtain a generalized pseudosound relation (at least at the lowest order), we then seek a modification to the homogeneous part by substituting the linear relationship into Eq. (58). This yields

$$-c_s^2\rho^{\infty}\nabla_{\eta}\cdot\boldsymbol{u}_1=c_s^2\boldsymbol{u}_1\cdot\nabla_{\eta}\rho^{\infty}. \tag{60}$$

On using Eq. (47) we obtain

$$\frac{\partial p^{*}}{\partial\tau'}=\frac{\gamma p_{\mathrm{SW}}}{\rho^{\infty}}\boldsymbol{u}_1\cdot\nabla_{\eta}\rho^{\infty}.$$

Because ρ^{∞} (together with p_{SW}, ρ_{SW}, and c_s^2) does not depend on the fast time τ', whereas $\boldsymbol{u}_1$ does, it yields the following expression for the "acoustic pressure"

$$p^{*}=\frac{\gamma p_{\mathrm{SW}}}{\rho^{\infty}}\int\boldsymbol{u}_1\cdot\nabla_{\eta}\rho^{\infty}d\tau'.$$

Note that p^{*} in the above expression serves only as a modification to its homogeneous counterpart. A generalized form of the acoustic component can then be written as

$$\mathcal{P}^{*}=p^{*}+\frac{\gamma p_{\mathrm{SW}}}{\rho^{\infty}}\int\boldsymbol{u}_1\cdot\nabla_{\eta}\rho^{\infty}d\tau'. \tag{61}$$

A generalized inhomogeneous pseudosound relation can then be expressed in a familiar form as

$$p^{\infty}+\mathcal{P}^{*}=c_s^2\rho_2. \tag{62}$$

Several interesting points emerge from this analysis. Compared to Zank and Matthaeus [5], the existence of the nonlinear part of the pseudosound relation is not directly due to solar wind background gradients [they are not present in Eq. (58)], but rather due to the inclusion of ϵ-order terms in the expansion of density fluctuations (namely ρ^{∞}). Were ρ^{∞} not included in the expansions (36) and (17), of the nearly incompressible model (ZM case), the last two terms in Eq. (58) would vanish and we would obtain only the usual pseudosound relation without the inhomogeneous part. However, as implied by Sec. IV, to self-consistently include solar wind background into the NI theory, the ϵ-order terms have to be included also. Therefore, the inhomogeneous part of the pseudosound relation (62) is due to the inclusion of the solar wind background, although indirectly.

For completeness (and demonstration of consistency of nearly incompressible theory) we note, that a similar form of the generalized inhomogeneous pseudosound relation can be derived, when in Eq. (60) we eliminate $\nabla_{\eta}\cdot\boldsymbol{u}_1$ by using Eq. (53). The pseudosound relation then can be written as

$$p^{\infty}+p^{*}=c_s^2\wp_2, \tag{63}$$

where

$$\wp_2=\rho_2+\frac{\rho_{\mathrm{SW}}}{\rho^{\infty}}\int\boldsymbol{u}_1\cdot\nabla_{\eta}\rho^{\infty}d\tau'. \tag{64}$$

This relation is of course consistent with Eq. (62), because the sound speed $c_s^2=\gamma p_{\mathrm{SW}}/\rho_{\mathrm{SW}}$, so the constants in front of the integrals are the same. The apparent inconsistency in sign has a very natural explanation. Both solutions are correct, since Eq. (58), being of second order, has two solutions. The two solutions represent backward and forward propagating waves with respect to the solar wind background. For a perturbation propagating in a direction opposite to the background gradient, we must observe increase of pressure and density and for a perturbation propagating in the same direction as the background gradient, we must observe a decrease. For that reason the pseudosound relation is written correctly with both signs as

$$p^{\infty}+p^{*}\pm\frac{\gamma p_{\mathrm{SW}}}{\rho^{\infty}}\int\boldsymbol{u}_1\cdot\nabla_{\eta}\rho^{\infty}d\tau'=c_s^2\rho_2. \tag{65}$$

VII. CONCLUSIONS

The main result of this paper is the derivation of a locally incompressible and nearly incompressible system of hydrodynamics equations in the presence of a large-scale inhomogeneous solar wind background (radially symmetric and in equilibrium). The theory was developed in the purly hydrodynamics regime under the assumption of low turbulent Mach number. The inclusion of large-scale inhomogeneities to nearly incompressible theory leads to several new analytical results, and we summarize them as follows.

(1) The presence of large-scale inhomogeneities modifies the leading-order incompressible description of solar wind, and is unlike the regular incompressible equations. For example, the divergence of the velocity fluctuations is nonsolenoidal and proportional to the large-scale gradients in solar wind. Large-scale gradients act as source terms and are responsible for inducing density fluctuations. On short-length scales, the system of equations in the leading-order incompressible description converge to the "usual" incompressible equations and therefore the term "locally incompressible" was introduced.

(2) The density fluctuations scale linearly with the Mach number $O(M_{s0})$, unlike the quadratic Mach number scaling $O(M_{s0}^2)$ of the homogeneous nearly incompressible theory of Zank and Matthaeus. This is consistent with Bhattacharjee *et al.* [17], who also found that $O(M_{s0})$ density fluctuations are generated as a result of coupling to the large-scale inhomogeneous magnetic field. We suggest that the linear Mach number scaling in density fluctuations is a typical feature of nearly incompressible inhomogeneous models.

(3) The special case, for which no large-scale inhomogeneous solar wind background flow is present, was considered and we showed, that both the locally incompressible and nearly incompressible equations converged to their corresponding homogeneous counterparts.

(4) Wave equations for the acoustic pressure (p^*) and velocity fluctuations ($\boldsymbol{u}_1$) propagating at the fast-time–short-wavelength scales with the sound speed c_s were derived. Unlike the regular homogeneous case, the speed of propagation varies spatially and depends on the solar wind background parameters $[c_s^2(r)=\gamma p_{\mathrm{SW}}/\rho_{\mathrm{SW}}]$.

(5) For fast-time scales (high frequencies), the pseudosound relation $p^\infty+p^*=c_s^2\rho_2$ relating density and pressure fluctuations was derived analytically, where again in contrast with the homogeneous case, the sound speed c_s^2 varies spatially.

(6) For slow-time scales (low frequencies), an implicit relationship between p^∞, p^*, and ρ_2 was derived and (at least at the lowest order) a generalized inhomogeneous pseudosound relation was obtained.

(7) An important outcome of this paper is that in understanding the physical meaning of expansion variables at different orders in ϵ, the validity of the sonic Mach number ($\epsilon=\gamma^{1/2}M_{s0}$) expansion of the fluid variables was clarified. For example, when $p'=\epsilon p_1+\epsilon^2(p^\infty+p^*)$, $\rho'=\epsilon\rho^\infty+\epsilon^2\rho_2$, $\boldsymbol{u}'=\boldsymbol{u}^\infty+\epsilon\boldsymbol{u}_1$, we found that the lowest order pressure p_1 must be excluded because it generates unbounded velocity oscillations. The pressure, density and velocity p^∞, ρ^∞, $\boldsymbol{u}^\infty$ vary only on slow-time scales and satisfy the locally incompressible equations. The higher order components p^*, ρ_2, $\boldsymbol{u}_1$ vary on both fast and slow-time scales and satisfy the nearly incompressible equations. The acoustic pressure p^* and velocity $\boldsymbol{u}_1$ are waves propagating with the sound speed, and p^∞, p^*, ρ_2 are related through the pseudosound relation. Without the inclusion of the large-scale inhomogeneous solar wind background, the ϵ-order term in density (ρ^∞) does not have to be included in the expansion series (unless we wish to include other effects such as heat conduction) as was done by Zank and Matthaeus.

Inclusion of heat conduction, together with the generalization of the inhomogeneous nearly incompressible theory to magnetohydrodynamics are under development and will be the topic of a subsequent paper. Extensive numerical simulations of the analytical results will also be addressed.

ACKNOWLEDGMENTS

This work has been supported in part by an NSF Grant No. ATM0428880 and ATM0317509.

APPENDIX

For convenience, we summarize the nearly incompressible equations in the presence of large scale solar wind inhomogeneities as follows.

1. Nearly incompressible (normalized)

Continuity:

$$\frac{\partial\rho_2}{\partial t}+\rho^\infty\boldsymbol{\nabla}\cdot\boldsymbol{u}_1+\boldsymbol{u}_1\cdot\boldsymbol{\nabla}\rho^\infty+(\boldsymbol{u}_{\mathrm{SW}}+\boldsymbol{u}^\infty)\cdot\boldsymbol{\nabla}\rho_2 = -\frac{1}{\epsilon}\rho_{\mathrm{SW}}\boldsymbol{\nabla}\cdot\boldsymbol{u}_1-\frac{\chi}{\epsilon}\boldsymbol{u}_1\cdot\boldsymbol{\nabla}_r\rho_{\mathrm{SW}}. \quad (A1)$$

Momentum:

$$\rho_{\mathrm{SW}}\frac{\partial\boldsymbol{u}_1}{\partial t}+\rho^\infty\frac{\partial\boldsymbol{u}^\infty}{\partial t}+\rho_{\mathrm{SW}}(\boldsymbol{u}_{\mathrm{SW}}+\boldsymbol{u}^\infty)\cdot\boldsymbol{\nabla}\boldsymbol{u}_1 +\rho^\infty(\boldsymbol{u}_{\mathrm{SW}}+\boldsymbol{u}^\infty)\cdot\boldsymbol{\nabla}\boldsymbol{u}^\infty+\rho_{\mathrm{SW}}\boldsymbol{u}_1\cdot\boldsymbol{\nabla}\boldsymbol{u}^\infty=-\frac{1}{\epsilon}\boldsymbol{\nabla}p^*. \quad (A2)$$

Energy equation:

$$\frac{\partial}{\partial t}(p^\infty+p^*)+(\boldsymbol{u}_{\mathrm{SW}}+\boldsymbol{u}^\infty)\cdot\boldsymbol{\nabla}(p^\infty+p^*)+\frac{1}{\epsilon}\gamma p_{\mathrm{SW}}\boldsymbol{\nabla}\cdot\boldsymbol{u}_1 = -\frac{\chi}{\epsilon}\boldsymbol{u}_1\cdot\boldsymbol{\nabla}_r p_{\mathrm{SW}}. \quad (A3)$$

It is useful to write down the nearly incompressible and locally incompressible equations with large scale inhomogeneities (solar wind background) in a non-normalized form.

2. Nearly incompressible (non-normalized)

Continuity:

$$\frac{\partial\rho_2}{\partial t}+\rho^\infty\boldsymbol{\nabla}\cdot\boldsymbol{u}_1+\boldsymbol{u}_1\cdot\boldsymbol{\nabla}\rho^\infty+(\boldsymbol{u}_{\mathrm{SW}}+\boldsymbol{u}^\infty)\cdot\boldsymbol{\nabla}\rho_2 = -\frac{1}{\gamma^{1/2}M_{s0}}\rho_{\mathrm{SW}}\boldsymbol{\nabla}\cdot\boldsymbol{u}_1-\frac{1}{\gamma^{1/2}M_{so}}\boldsymbol{u}_1\cdot\boldsymbol{\nabla}_r\rho_{\mathrm{SW}}. \quad (A4)$$

Momentum:

$$\rho_{\mathrm{SW}}\frac{\partial\boldsymbol{u}_1}{\partial t}+\rho^\infty\frac{\partial\boldsymbol{u}^\infty}{\partial t}+\rho_{\mathrm{SW}}(\boldsymbol{u}_{\mathrm{SW}}+\boldsymbol{u}^\infty)\cdot\boldsymbol{\nabla}\boldsymbol{u}_1+\rho^\infty(\boldsymbol{u}_{\mathrm{SW}} +\boldsymbol{u}^\infty)\cdot\boldsymbol{\nabla}\boldsymbol{u}^\infty+\rho_{\mathrm{SW}}\boldsymbol{u}_1\cdot\boldsymbol{\nabla}\boldsymbol{u}^\infty=-\gamma^{1/2}M_{s0}\boldsymbol{\nabla}p^*. \quad (A5)$$

Energy:

$$\frac{\partial}{\partial t}(p^\infty+p^*)+(\boldsymbol{u}_{\mathrm{SW}}+\boldsymbol{u}^\infty)\cdot\boldsymbol{\nabla}(p^\infty+p^*)+\frac{\gamma^{1/2}}{M_{s0}}p_{\mathrm{SW}}\boldsymbol{\nabla}\cdot\boldsymbol{u}_1 = -\frac{1}{\gamma^{1/2}M_{s0}}\boldsymbol{u}_1\cdot\boldsymbol{\nabla}_r p_{\mathrm{SW}}. \quad (A6)$$

The non-normalized expansion for the fluctuating quantities is the same as Eq. (37). Note, that the coefficient $\gamma^{1/2}p_{\mathrm{SW}}/M_{s0}$ in front of $\boldsymbol{\nabla}\cdot\boldsymbol{u}_1$ in the energy equation can be rewritten as $\rho_{\mathrm{SW}}c_s^2/(\gamma^{1/2}M_{s0})$ which, in the absence of a solar wind background, becomes $\rho_0c_{s0}^2/(\gamma^{1/2}M_{s0})$; this differs from ZM in their final non-normalized energy equation, who erroneously had $\gamma^{3/2}$.

3. Locally incompressible (non-normalized)

$$\nabla \cdot \boldsymbol{u}^{\infty} = -\frac{1}{\rho_{\mathrm{SW}}} \boldsymbol{u}^{\infty} \cdot \nabla_r \rho_{\mathrm{SW}}, \tag{A7}$$

$$\rho_{\mathrm{SW}} \frac{\partial \boldsymbol{u}^{\infty}}{\partial t} + \rho_{\mathrm{SW}} (\boldsymbol{u}_{\mathrm{SW}} + \boldsymbol{u}^{\infty}) \cdot \nabla \boldsymbol{u}^{\infty} = -\gamma M_{s0}^2 \nabla p^{\infty} - \rho_{\mathrm{SW}} \boldsymbol{u}^{\infty} \cdot \nabla_r \boldsymbol{u}_{\mathrm{SW}}; \tag{A8}$$

$$\frac{\partial \rho^{\infty}}{\partial t} + (\boldsymbol{u}_{\mathrm{SW}} + \boldsymbol{u}^{\infty}) \cdot \nabla \rho^{\infty} = -\rho^{\infty} \nabla_r \cdot \boldsymbol{u}_{\mathrm{SW}} + \frac{\rho^{\infty}}{\rho_{\mathrm{SW}}} \boldsymbol{u}^{\infty} \cdot \nabla_r \rho_{\mathrm{SW}}. \tag{A9}$$

4. ZANK and MATTHAEUS nearly incompressible (non-normalized)

$$\frac{\partial \rho_2}{\partial t} + \boldsymbol{u}^{\infty} \cdot \nabla \rho_2 + \frac{\rho_0}{\gamma^{1/2} M_s} \nabla \cdot \boldsymbol{u}_1 = 0, \tag{A10}$$

$$\frac{\partial \boldsymbol{u}_1}{\partial t} + \boldsymbol{u}^{\infty} \cdot \nabla \boldsymbol{u}_1 + \boldsymbol{u}_1 \cdot \nabla \boldsymbol{u}^{\infty} = -\frac{\gamma^{1/2} M_s}{\rho_0} \nabla p^*, \tag{A11}$$

$$\frac{\partial}{\partial t}(p^{\infty} + p^*) + \boldsymbol{u}^{\infty} \cdot \nabla (p^{\infty} + p^*) + \frac{\rho_0 c_s^2}{\gamma^{1/2} M_s} \nabla \cdot \boldsymbol{u}_1 = 0. \tag{A12}$$

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